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Nombre__________________________________Curso: A-___

Pregunta 1

Para la función $f(x,y) = \begin{cases} -\dfrac{3xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

a) Calcule $f_x(x,y).$, $f_y(x,y)$ para todo $(x,y) \in \mathbb{R}^2$.

Solución

$$f_x(x,y) = -\frac{3y(x^2+y^2) - 3xy \cdot 2x}{(x^2+y^2)^2} = \frac{-3x^2y - 3y^3 + 6x^2y}{(x^2+y^2)^2}$$

$$\therefore f_x(x,y) = \frac{3x^2y - 3y^3}{(x^2+y^2)^2} \text{ si } (x,y) \neq (0,0)$$

Además

$$f_x(0,0) = \lim_{h \longrightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \longrightarrow 0} \frac{0-0}{h} = \lim_{h \longrightarrow 0} 0 = 0$$

Por lo tanto

$$f_x(x,y) = \begin{cases} \dfrac{3x^2y - 3y^3}{(x^2+y^2)^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$$

Análogamente por simetría de la expresión se tiene

$$f_y(x,y) = \begin{cases} \dfrac{3xy^2 - 3x^3}{(x^2+y^2)^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$$

b) Pruebe que las derivadas parciales primeras son discontinuas en $(0,0)$.

Solución

Trataremos solo el caso de la discontinuidad de $f_x(x,y)$ en $(0,0)$, la otra es similar.

Supongamos que existe $\displaystyle\lim_{(x,y) \longrightarrow (0,0)} \frac{3x^2y - 3y^3}{(x^2+y^2)^2}$, usando límites iterados tenemos

$$\lim_{y \longrightarrow 0} \left(\lim_{x \longrightarrow 0} \frac{3x^2y - 3y^3}{(x^2+y^2)^2} \right) = \lim_{x \longrightarrow 0} \left(\frac{-3y^3}{y^4} \right) = \lim_{y \longrightarrow 0} \left(\frac{-3}{y} \right) = -\infty$$

Luego, no existe límite en $(0,0)$ y la función $f_x(x,y)$ no es continua en $(0,0)$.

c) Determine la diferencial total de f en $(0,0)$.

Solución

$$df(0,0) = \frac{\partial f}{\partial x}(0,0)dx + \frac{\partial f}{\partial y}(0,0)dx = 0 \cdot dx + 0 \cdot dy = 0$$

d) Pruebe que f no es diferenciable en $(0,0)$.

Solución

$f(h,k)-f(0,0)=df(0,0)+\alpha(h,k)\sqrt{x^2+y^2}$

$$\Longrightarrow \alpha(h,k)=\frac{-3hk}{(h^2+k^2)^{\frac{3}{2}}}=\frac{-3h^2}{h^3(1+1)^{\frac{3}{2}}},$$

$$si \quad k=h \quad se \quad tiene \quad \alpha(h,h)=\frac{-3hh}{(h^2+h^2)^{\frac{3}{2}}}$$

$$y \quad \lim_{h\longrightarrow 0}\frac{3}{2^{\frac{3}{2}}h}\neq 0$$

1. $\therefore f$ no es diferenciable

e) Determine en qué dirección la derivada direccional en $(0,0)$ no existe.

Solución

$$\frac{\partial f}{\partial \widehat{a}}=\lim_{h\longrightarrow 0}\frac{f(a_1h,a_2h)-f(0,0)}{h}=\lim_{h\longrightarrow 0}\frac{-3a_1a_2h^2}{h(a_1^2+a_2^2)h^2}=\infty$$

a menos que $\widehat{a}=(1,0)$ $ó$ $(0,1)$

Pregunta 2

Considere la ecuación

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0 \qquad ,z=f(x,y)$$

donde f es una función real de dos variables de clase C^2.Transformar la ecuación diferencial con el cambio

$u=xy,v=\dfrac{x}{y}$

Solución

$$\left.\begin{array}{l} u=xy \\ v=\dfrac{x}{y}\end{array}\right\}\Longrightarrow \begin{array}{l} u=(vy)\,y \\ x=vy \end{array} \begin{array}{l}\Longrightarrow \\ \Longrightarrow\end{array} \begin{array}{l} y^2=\dfrac{u}{v} \\ x=v\sqrt{\dfrac{u}{v}}\end{array} \begin{array}{l}\Longrightarrow \\ \Longrightarrow\end{array} \begin{array}{l} y=\left(\dfrac{u}{v}\right)^{\frac{1}{2}} \\ x=(uv)^{\frac{1}{2}}\end{array}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\Longrightarrow \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}u_x+\frac{\partial z}{\partial v}v_x$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial u^2}(u_x)^2+\frac{\partial^2 z}{\partial v\partial u}(u_x)(v_x)+\frac{\partial z}{\partial u}u_{xx}+\frac{\partial^2 z}{\partial u\partial v}(v_x)(u_x)+\frac{\partial^2 z}{\partial v^2}(v_x)^2+\frac{\partial z}{\partial v}v_{xx}$$

$$u_x=y=\left(\frac{u}{v}\right)^{\frac{1}{2}}$$

$$v_x=\frac{1}{y}=\left(\frac{v}{u}\right)^{\frac{1}{2}}$$

$$u_{xx}=0$$

$$v_{xx}=0$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2}\left(\frac{u}{v}\right) + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\left(\frac{v}{u}\right)^2$$

$$x^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2}u^2 + 2\frac{\partial^2 z}{\partial u \partial v}uv + \frac{\partial^2 z}{\partial v^2}v^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \implies \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}u_y + \frac{\partial z}{\partial v}v_y$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial u^2}(u_y)^2 + \frac{\partial^2 z}{\partial v \partial u}(u_y)(v_y) + \frac{\partial z}{\partial u}u_{yy} + \frac{\partial^2 z}{\partial u \partial v}(v_y)(u_y) + \frac{\partial^2 z}{\partial v^2}(v_y)^2 + \frac{\partial z}{\partial v}v_{yy}$$

$$u_y = x = (uv)^{\frac{1}{2}}$$

$$v_y = -\frac{x}{y^2} = \frac{-(uv)^{\frac{1}{2}}}{\frac{u}{v}} = -v(\tfrac{v}{u})^{\frac{1}{2}}$$

$$u_{yy} = 0$$

$$v_{yy} = -\frac{2\,(uv)^{\frac{1}{2}}}{\left(\frac{u}{v}\right)^{\frac{3}{2}}} = -2\frac{v^2}{u}$$

$$\therefore v_{yy} = -2\frac{v^2}{u}$$

Ahora

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2}(uv) - 2\frac{\partial^2 z}{\partial v \partial u}v^2 + \frac{\partial^2 z}{\partial v^2}\frac{v^3}{u} - 2\frac{\partial z}{\partial v}\frac{v^2}{u}$$

$$y^2\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2}u^2 - 2\frac{\partial^2 z}{\partial v \partial u}uv + \frac{\partial^2 z}{\partial v^2}v^2 - 2\frac{\partial z}{\partial v}v$$

Restando

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 4\frac{\partial^2 z}{\partial u \partial v}uv + 2\frac{\partial z}{\partial v}v$$

Se obtiene la ecuación

$$4\frac{\partial^2 z}{\partial u \partial v}uv + 2\frac{\partial z}{\partial v}v = 0$$

Pregunta 3

El plano $x + y + z = 24$ cruza al paraboloide $z = x^2 + y^2$, generando en la intersección una elipse. Calcule el punto más alto y más bajo de la elipse respecto del plano XY

Solución

$F(x, y, z, \lambda_1, \lambda_2) = z + \lambda_1\,(x + y + z - 24) + \lambda_2\left(z - x^2 - y^2\right)$

$$F_x = \lambda_1 - 2x\lambda_2 = 0 \implies x = \frac{\lambda_1}{2\lambda_2}$$

$F_y = \lambda_1 - 2y\lambda_2 = 0 \implies y = \dfrac{\lambda_1}{2\lambda_2}$

$F_z = 1 + \lambda_1 + \lambda_2 = 0 \implies \lambda_1 + \lambda_2 = -1 \implies \lambda_2 = -1 - \lambda_1$

$F_{\lambda_1} = x + y + z - 24 = 0 \implies z = 24 - x - y = 24 - \dfrac{\lambda_1}{\lambda_2}$

$F_{\lambda_2} = z - x^2 - y^2 = 0 \implies 24 - \dfrac{\lambda_1}{\lambda_2} - \left(\dfrac{\lambda_1}{2\lambda_2}\right)^2 - \left(\dfrac{\lambda_1}{2\lambda_2}\right)^2 = 0$

$\left(\dfrac{\lambda_1}{\lambda_2}\right)^2 + 2\left(\dfrac{\lambda_1}{\lambda_2}\right) - 48 = 0$

Sea $a = \dfrac{\lambda_1}{\lambda_2}$

$a^2 + 2a - 48 = 0 \iff (a+8)(a-6) = 0$

$\dfrac{\lambda_1}{\lambda_2} = -8 \implies \dfrac{\lambda_1}{-1-\lambda_1} = -8 \implies \lambda_1 = 8 + 8\lambda_1 \implies \lambda_1 = -\dfrac{8}{7}$

$\dfrac{\lambda_1}{\lambda_2} = 6 \implies \dfrac{\lambda_1}{-1-\lambda_1} = 6 \implies \lambda_1 = -6 - 6\lambda_1 \implies \lambda_1 = -\dfrac{6}{7}$

$\lambda_2 = -1 + \dfrac{8}{7} = \dfrac{1}{7}$

$\lambda_2 = -1 + \dfrac{6}{7} = -\dfrac{1}{7}$

Así
I) $\lambda_1 = -\dfrac{8}{7}$ y $\lambda_2 = \dfrac{1}{7}$

II) $\lambda_1 = -\dfrac{6}{7}$ y $\lambda_2 = -\dfrac{1}{7}$

$$y = x = \frac{-\frac{8}{7}}{2\frac{1}{7}} = -4 \implies z = 24 + 4 + 4 = 32$$

$$y = x = \frac{-\frac{6}{7}}{2\left(-\frac{1}{7}\right)} = 3 \implies z = 24 - 3 - 3 = 18$$

De lo anterior se deduce que $(-4, -4, 32)$ es el punto más alto de la elipse y $(3, 3, 18)$ es el punto más bajo de tal elipse.