U. DE SANTIAGO DE CHILE FAC. DE CIENCIA DEP. DE MATEMÁTICA Y C.C.

PRIMERA PRUEBA CÁLCULO AVANZADO 10007
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Pregunta 1

Utilizando serie de Fourier para $f(x) = e^x$ en $[-\pi, \pi]$. Demuestre que

a) $e^x = \dfrac{asenh\pi}{\pi}\left[\dfrac{1}{a} + \sum_1^\infty \left(\dfrac{(-1)^n}{1+n^2}\cos(nx) - \dfrac{n(-1)^n}{1+n^2}sen(nx)\right)\right]$ para algún $a \in \mathbb{R}$

b) $\sum_1^\infty \dfrac{1}{1+n^2} = b\left(\dfrac{\pi - \tanh\pi}{\tanh\pi}\right)$, para algún $b \in \mathbb{R}$

Solución
a)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{2}{\pi}\left(\frac{e^\pi - e^{-\pi}}{2}\right) = \frac{2senh(\pi)}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)dx = \frac{1}{\pi}\left[\frac{e^x\,(nsen(nx) + \cos(nx))}{n^2+1}\right]\Big|_{-\pi}^{\pi}$$

$$a_n = \frac{2(-1)^n senh(\pi)}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x sen(nx)dx = \frac{1}{\pi}\left[\frac{e^x\,(-n\cos(nx) + sen(nx))}{n^2+1}\right]\Big|_{-\pi}^{\pi}$$

$$b_n = \frac{-2n(-1)^n senh(\pi)}{\pi(n^2+1)}$$

luego

$$e^x = \frac{2senh(\pi)}{2\pi} + \sum_1^\infty \left(\frac{2(-1)^n senh(\pi)}{\pi(n^2+1)}cos(nx) - \frac{2n(-1)^n senh(\pi)}{\pi(n^2+1)}sen(nx)\right)$$

$$e^x = \frac{2senh(\pi)}{\pi}\left[\frac{1}{2} + \sum_1^\infty \left(\frac{(-1)^n}{n^2+1}cos(nx) - \frac{n(-1)^n}{n^2+1)}sen(nx)\right)\right]$$

$\therefore a = 2$

b)

Si $x = \pi$

$$S(\pi) = \frac{f(\pi^+) + f(\pi^-)}{2} = \frac{e^{\pi} + e^{-\pi}}{2} = \cosh(\pi)$$

luego

$$\cosh(\pi) = \frac{2senh(\pi)}{\pi}\left[\frac{1}{2} + \sum_{1}^{\infty}\left(\frac{(-1)^n \cdot (-1)^n}{n^2+1}\right)\right]$$

$$\frac{\pi}{2}\frac{\cosh(\pi)}{\sinh(\pi)} = \frac{1}{2} + \sum_{1}^{\infty}\left(\frac{1}{n^2+1}\right)$$

$$\frac{\pi}{2}\frac{\cosh(\pi)}{\sinh(\pi)} - \frac{1}{2} = \sum_{1}^{\infty}\left(\frac{1}{n^2+1}\right)$$

$$\frac{1}{2}\left[\frac{\pi\cosh(\pi)}{\sinh(\pi)} - 1\right] = \sum_{1}^{\infty}\left(\frac{1}{n^2+1}\right)$$

$$\frac{1}{2}\left[\frac{\pi}{\tanh(\pi)} - 1\right] = \sum_{1}^{\infty}\left(\frac{1}{n^2+1}\right)$$

$$\frac{1}{2}\left[\frac{\pi - \tanh(\pi)}{\tanh(\pi)}\right] = \sum_{1}^{\infty}\left(\frac{1}{n^2+1}\right)$$

$\therefore b = \frac{1}{2}$

Pregunta 2

Para la curva C dada por $\overrightarrow{r}(t) = \left(t - sent, 1 - \cos t, 4sen\left(\frac{t}{2}\right)\right)$, $t \in [0, 2\pi]$.

a) Verifique si la curva es regular

b) Calcular su longitud L.

c) Determinar la ecuación de la recta tangente en el punto $(\pi, 2, 4)$

Solución

a) A partir de $\overrightarrow{r}(t) = \left(t - sent, 1 - \cos t, 4sen\left(\frac{t}{2}\right)\right)$ tenemos

$$\overrightarrow{r}'(t) = \left(1 - \cos t, sent, 2\cos\left(\frac{t}{2}\right)\right) \neq \overrightarrow{0} \ \forall t \in [0.2\pi]$$

luego la curva es regular

b) Entonces: $\|\overrightarrow{r}'(t)\| = \sqrt{(1-\cos t)^2 + sen^2 t, 4\cos^2\left(\dfrac{t}{2}\right)}$

$$\|\overrightarrow{r}'(t)\| = \sqrt{1 - 2\cos t + \cos^2 t + sen^2 t + 4\cos^2\left(\frac{t}{2}\right)}$$

$$\|\overrightarrow{r}'(t)\| = 2$$

luego $L(\overrightarrow{r}) = \displaystyle\int_0^{2\pi} 2dt = 2t\Big|_0^{2\pi} = 4\pi$

c) $\overrightarrow{r}(t_0) = \left(t_0 - sent_0, 1 - \cos t_0, 4sen\left(\dfrac{t_0}{2}\right)\right) = (\pi, 2, 4)$

$$4sen\left(\frac{t_0}{2}\right) = 4 \Rightarrow sen\left(\frac{t_0}{2}\right) = 1 \Rightarrow t_0 = \pi$$

$$\overrightarrow{r}(\pi) = (\pi, 2, 4)$$

$$\overrightarrow{r}'(\pi) = (2, 0, 0)$$

Entonces la ecuación de la recta tangente en el punto $\overrightarrow{r}(\pi)$ es

$\overrightarrow{r}(\alpha) = \overrightarrow{r}(\pi) + \alpha\overrightarrow{r}'(\pi)$

$\overrightarrow{r}(\alpha) = (\pi, 2, 4) + \alpha(2, 0, 0)$

$y = 2$
$z = 4$

Pregunta 3

Para la curva C descrita por $\overrightarrow{r}(t) = (2t, t^2, \ln t)$, $t \in]0, +\infty[$.

a) Determinar los vectores $\widehat{\mathrm{T}}$, $\widehat{\mathrm{N}}$ y $\widehat{\mathrm{B}}$.

b) Calcule la curvatura en cualquier punto de C .

c) Obtenga el punto de la curva en que la curvatura es máxima.

Solución

a)

$c(t) = \left(2t, t^2, \ln t\right),\ t \in]0, +\infty[$

$$c'(t) = \left(2, 2t, \frac{1}{t}\right)$$

$$\|c'(t)\| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = \sqrt{\frac{4t^2 + 4t^4 + 1}{t^2}}$$

$$\|c'(t)\| = \sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{2t^2+1}{t}$$

$$\widehat{T} = \frac{\frac{dc}{dt}}{\left\|\frac{dc}{dt}\right\|} \Rightarrow \widehat{T} = \frac{t}{2t^2+1}\left(2, 2t, \frac{1}{t}\right)$$

$$c''(t) = \left(0, 2, -\frac{1}{t^2}\right)$$

$$c'(t) \times c''(t) = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 2 & 2t & \frac{1}{t} \\ 0 & 2 & -\frac{1}{t^2} \end{vmatrix} = \left(-\tfrac{4}{t}\right)\widehat{i} + \tfrac{2}{t^2}\widehat{j} + 4\widehat{k}$$

$$\|c'(t) \times c''(t)\| = \left\|\frac{2}{t^2}(-2t, 1, 2t^2)\right\| = \tfrac{2}{t^2}\sqrt{4t^2 + 1 + 4t^4)}$$

$$\therefore \|c'(t) \times c''(t)\| = \frac{2}{t^2}\sqrt{(2t^2+1)^2} = \tfrac{2}{t^2}(2t^2+1)$$

Asi $\widehat{B} = \dfrac{c'(t) \times c''(t)}{\|c'(t) \times c''(t)\|} \Rightarrow \widehat{B} = \dfrac{t^2}{2(2t^2+1)}\left(-\tfrac{4}{t}, \tfrac{2}{t^2}, 4\right)$

simplificando $\widehat{B} = \dfrac{1}{2t^2+1}\left(-2t, 1, 2t^2\right)$

$$\widehat{N} = \widehat{B} \times \widehat{T} = \frac{1}{(2t^2+1)^2}\begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ -2t & 1 & 2t^2 \\ 2t & 2t^2 & \end{vmatrix} = \frac{1}{(2t^2+1)^2}\left(1 - 4t^4, 4t^3 + 2t, -(4t^3 + 2t)\right)$$

$$\widehat{N} = \frac{1}{(2t^2+1)^2}\left(\left(1 - 2t^2\right)\left(1 + 2t^2\right), 2t\left(2t^2+1\right), -2t\left(2t^2+1\right)\right)$$

$$\widehat{N} = \frac{1}{2t^2+1}\left(1 - 2t^2, 2t, -2t\right)$$

b)

$$K(t) = \|c''(s)\| = \left\| \frac{d\widehat{T}}{ds} \right\| = \left\| \frac{d\widehat{T}}{dt} \cdot \frac{dt}{ds} \right\| = \frac{\left\| \frac{d\widehat{T}}{dt} \right\|}{\|c'(s)\|}$$

$$\widehat{T} = \frac{1}{2t^2+1} \left(2t, 2t^2, 1\right) = \left(\frac{2t}{2t^2+1}, \frac{2t^2}{2t^2+1}, \frac{1}{2t^2+1} \right)$$

$$\frac{d\widehat{T}}{dt} = \left(\frac{2(2t^2+1) - 8t^2}{(2t^2+1)^2}, \frac{4t(2t^2+1) - 8t^3}{(2t^2+1)^2}, \frac{-4t}{(2t^2+1)^2} \right)$$

$$\frac{d\widehat{T}}{dt} = \frac{2}{(2t^2+1)^2} \left(1 - 2t^2, 2t, -2t\right)$$

$$\left\| \frac{d\widehat{T}}{dt} \right\| = \frac{2}{(2t^2+1)^2} \sqrt{1 + 4t^4 + 4t^2} = \frac{2}{(2t^2+1)^2} \sqrt{(2t^2+1)^2}$$

$$\left\| \frac{d\widehat{T}}{dt} \right\| = \frac{2}{(2t^2+1)}$$

pero $\|c'(t)\| = \dfrac{2t^2+1}{t}$

Por lo tanto $K(t) = \dfrac{2t}{(2t^2+1)^2}$

c)

$$\frac{K(t)}{dt} = \frac{2\left(2t^2+1\right)^2 - 16t^2\left(2t^2+1\right)}{(2t^2+1)^4} = \frac{2(1-6t^2)}{(1+2t^2)^3}$$

Punto crítico: $t = \dfrac{1}{\sqrt{6}}$

Si $t < \dfrac{1}{\sqrt{6}}$ entonces $K'(t) > 0$ y

si $t > \dfrac{1}{\sqrt{6}}$ entonces $K'(t) < 0$

Luego en $t = \dfrac{1}{\sqrt{6}}$ la curvatura es máxima y ocurre

en $c\left(\frac{1}{\sqrt{6}}\right) = \left(\dfrac{2}{\sqrt{6}}, \dfrac{1}{6}, \ln \dfrac{1}{\sqrt{6}} \right)$