DIFERENCIABILIDAD

GUÍA 2

EJERCICIOS RESUELTOS

1.Demuestre que la función f definida por $f(x,y)=2x^2+3xy$ es diferenciable en todo punto de $\Re^2$.

Solución.

Sea (x,y) punto cualquiera de $\Re^2$ y (h,k) vector de $\Re^2$.

$$f(x+h,y+k)=2(x+h)^2+3(x+h)(y+k)=2x^2+4xh+2h^2+3xy+3xk+3yh+3hk$$

Ordenando adecuadamente se tiene:

$$f(x+h,y+k)=2x^2+3xy+(4x+3y)k+3xk+2h^2+3hk$$

entonces $r(h,k)=2h^2+3hk$

$$\left|\frac{2h^2+3hk}{\sqrt{h^2+k^2}}\right|\le\left|\frac{2h^2+3(h^2+k^2)+2k^2}{\sqrt{h^2+k^2}}\right|\le 5\sqrt{x^2+h^2}$$

$$\lim_{(h,k)\to(0,0)}\sqrt{h^2+k^2}=0 \quad\Rightarrow\quad \therefore \lim_{(h,k)\to(0,0)}\frac{2h^2+3hk}{\sqrt{h^2+k^2}}=0$$

2.Si $u=x^2y+y^2z+z^2x$ y si z se define implícitamente como una función de x e y por $x^2+yz+z^3=0$, calcule $\dfrac{\partial u}{\partial x}$, cuando u es considerada como una función de x e y solamente.

Solución.

Sean $f(x,y,z)=x^2y+y^2z+z^2x$, $g(x,y,z)=x^2+yz+z^2$, y $z=\phi(x,y)$ es la función determinada por $g(x,y,z)=0$ y $u=\psi(x,y)=f(x,y,\phi(x,y))$.

Entonces derivando respecto de x:

$$\begin{aligned}\frac{\partial u}{\partial x} = \psi_x(x,y) &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial \phi}{\partial x} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\left(-\frac{g_x}{g_z}\right) \\ &= 2xy + z^2\left(y^2 + 2zx\right)\left(\frac{2x}{y+3z^2}\right) \\ &= \frac{6xyz^2 - 4x^2z + yz^2 + 3z^4}{y+3z^2}\end{aligned}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{6xyz^2 - 4x^2z + yz^2 + 3z^4}{y+3z^2}$$

3.Calcular la derivada direccional de $f(x,y,z) = 2x^2 - z^2 - y^2$ en (1,2,2) hacia (4,5,0)

Solución

$$\vec{u} = \frac{(4,5,0)-(1,2,2)}{\|(4,5,0)-(1,2,2)\|} = \frac{(3,3,-2)}{\|(3,3,-2)\|} = \frac{1}{\sqrt{22}}(3,3,-2)$$

$$\nabla f(x,y,z) = (4x,-2y,-2z) \Rightarrow \nabla f(1,2,2) = (4,-4,-4)$$

$$\begin{aligned}D_u f(1,2,2) = \nabla f(1,2,2)\cdot u &= (4,-4,-4)\cdot\frac{1}{\sqrt{22}}(3,3,-2) \\ &= \frac{1}{\sqrt{22}}(12-12+8) = \frac{8}{\sqrt{22}}\end{aligned}$$

Luego $D_u f(1,2,2) = \dfrac{8}{\sqrt{22}}$.

4.Para la superficie dada por $z = 20 - 4x^2 - y^2$

a) En $P_0 = (2,-3)$ determinar la dirección y el valor del máximo crecimiento de $f(x,y)$

b) Obtener el punto P_1 de S en el cual el plano tangente a S determina trazos iguales en los ejes coordenados. Escribir la ecuación de este plano tangente.

Solución.

a) $\nabla f(x,y)=(8x,-2y)$ y $\nabla f(x,y)=(-16,6)$

dirección $\vec{u}=\dfrac{(-16,6)}{\sqrt{(-16)^2+6^2}}=\dfrac{(-16,6)}{\sqrt{292}}$ de máximo crecimiento

$\|\nabla f(2,-3)\|=\sqrt{292}$ valor de máximo crecimiento.

b)La normal $\vec{n}$ a S es $\vec{n}=(-8x,-2y,-1)=\lambda(1,1,1)$

entonces $-8x=\lambda$, $-2y=\lambda$, $-1=\lambda$ por lo que $x=\dfrac{1}{8}$, $y=\dfrac{1}{2}$,

$z=20-4\left(\dfrac{1}{8}\right)^2-\left(\dfrac{1}{2}\right)^2=\dfrac{315}{16}$

Así que el punto pedido es $P_1=\left(\dfrac{1}{8},\dfrac{1}{2},\dfrac{315}{16}\right)$

La ecuación del plano tangente a S en P_1 es $\left(x-\dfrac{1}{8}\right)+\left(y-\dfrac{1}{2}\right)+\left(z-\dfrac{315}{16}\right)=0$

o $x+y+z=\dfrac{315}{16}$.

5.Para la transformación $u(x,y)=\dfrac{1-x^2-y^2}{x^2+(1+y)^2}$, $v(x,y)=\dfrac{2x}{x^2+(1+y)^2}$ con $(x,y)\neq(0,-1)$

a) Verificar que : $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$; $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

b) Calcular $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$; $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}$

Solución.

a)Operando directamente

$$\frac{\partial u}{\partial x}=\frac{-2x(x^2+(1+y)^2)-2x(1-x^2-y^2)}{\left[x^2+(1+y)^2\right]^2}=\frac{-2x^3-2x-4xy-2xy^2-2x+2x^3+2xy^2}{\left[x^2+(1+y)^2\right]^2}$$

$$=\frac{-4x-4xy}{\left[x^2+(1+y)^2\right]^2}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=\frac{-2y(x^2+(1+y)^2)-(1-x^2-y^2)2(1+y)}{\left[x^2+(1+y)^2\right]^2}$$

$$\frac{\partial u}{\partial y}=\frac{-2x^2y-2y-4y^2-2y^3-2+2x^2+2y^2-2y+2x^2y+2y^3}{\left[x^2+(1+y)^2\right]^2}$$

$$\frac{\partial u}{\partial y}=\frac{2x^2-2y^2-4y-2}{\left[x^2+(1+y)^2\right]^2}=-\frac{2y^2-2x^2+4y+2}{\left[x^2+(1+y)^2\right]^2}=-\frac{\partial v}{\partial x}$$

b)Usando las igualdades de a)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

por continuidad

$$=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=0$$

6.Escribir la ecuación de Laplace en coordenadas polares.

Ecuación de Laplace: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

Solución.

Donde $x=r\cos\theta$, $y=r\sin\theta$

Derivando:

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\Rightarrow\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\Rightarrow\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}r\cos\theta$$

De el sistema

$$\begin{cases}\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta=\frac{\partial u}{\partial r}\\ -\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta=\frac{\partial u}{\partial \theta}\end{cases}$$

Resolviendo para $\frac{\partial u}{\partial x}$ y $\frac{\partial u}{\partial y}$ se tiene:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\sin\theta}{r}$$
$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} - \frac{\partial^2 u}{\partial r\partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r} - \frac{\partial^2 u}{\partial \theta\partial r}\frac{\sin\theta\cos\theta}{r} +$$
$$\frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\sin\theta + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\cos\theta}{r}$$
$$= \frac{\partial^2 u}{\partial r^2}\sin^2\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 u}{\partial r\partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r} + \frac{\partial^2 u}{\partial \theta\partial r}\frac{\sin\theta\cos\theta}{r} -$$
$$\frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos^2\theta}{r^2}$$

Sumando se tiene

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

7. Establecer para qué ángulos *x, y, z* cuya suma es $\frac{\pi}{2}$, o sea $x + y + z = \frac{\pi}{2}$ se cumple la desigualdad:

$$\sin x \sin y \sin z \le \frac{1}{8}$$

Solución.

Sea $f(x,y,z) = \sin x \sin y \sin z$ la cual se estudiará extremo con condición $g(x,y,z) = x + y + z - \frac{\pi}{2} = 0$

Para obtener los puntos críticos definimos

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$$

Obteniéndose el sistema:

$$L_x = \cos x \sin y \sin z + \lambda = 0 \quad \Rightarrow \quad \lambda = -\cos x \sin y \sin z$$
$$L_y = \sin x \cos y \sin z + \lambda = 0 \quad \Rightarrow \quad \lambda = -\sin x \cos y \sin z$$
$$L_z = \sin x \sin y \cos z + \lambda = 0 \quad \Rightarrow \quad \lambda = -\sin x \sin y \cos z$$
$$L_\lambda = x + y + z - \frac{\pi}{2} = 0$$
$$\Rightarrow \sin x \cos y - \cos x \sin y = 0 \qquad \sin(x-y) = 0$$
$$\Rightarrow$$
$$\sin y \cos z - \cos y \sin z = 0 \qquad \sin(y-z) = 0$$

De donde se concluye que $x = y = z$, por lo cual $x + y + z = \frac{\pi}{2} \Rightarrow x = y = z = \frac{\pi}{6}$ o sea $P\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right)$ es punto crítico.

Condición suficiente : de $z = z(x,y) = \frac{\pi}{2} - x - y$ se tiene $z_x = -1, \quad z_y = -1$

$$f_x = \sin y(\cos x \sin z + \sin z \cos z \cdot (-1)) = \sin t \sin(z-x)$$
$$f_y = \sin x(\cos y \sin z - \sin y \cos z) = \sin x \sin(z-y)$$
$$f_{xx} = \sin y \cos(z-x) \cdot (-1-1) = -2 \sin y \cos(z-x)$$
$$f_{yy} = \sin x \cos(z-y) \cdot (-1-1) = -2 \sin x \cos(z-y)$$
$$f_{xy} = \cos y \sin(z-x) + \sin y \cos(z-x)(-1) = \sin(z-x-y)$$

El Hessiano entonces en P_0 es:

$$H(P_0) = \begin{vmatrix} -2 \cdot \frac{1}{2} \cdot 1 & -\frac{1}{2} \\ -\frac{1}{2} & -2 \cdot \frac{1}{2} \cdot 1 \end{vmatrix} = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

$f_{xx}(P_0) = -1 < 0 \Rightarrow f(P_0)$ es valor máximo

Por lo cual se cumple que:

$$\sin x \sin y \sin z \leq \frac{1}{8}$$