UNIVERSIDAD DE SANTIAGO DE CHILE FACULTAD DE CIENCIA

DEPARTAMENTO DE MATEMATICA Y C.C.

PRIMERA PRUEBA DE CÁLCULO AVANZADO PARA INGENIERÍA 10121

Ingeniería Civil Segundo Semestre 2013

(10/05/2013)

Pregunta 1:

Sea $\mathbf{f(x)} = \begin{cases} -\cos x, & \text{si} \quad -\pi \le x < 0 \\ 0, & \text{si} \quad x = 0 \\ \cos x, & \text{si} \quad 0 < x \le \pi \end{cases}$, extendida a IR como función de período 2π.

a) Obtener la serie de Fourier de f.

b) Usar a) para calcular $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k-1)}{4(2k-1)^2 - 1}$.

c) Usar a) para calcular $\sum_{k=1}^{\infty} \frac{k^2}{(4k^2-1)^2}$

Solución:

Como f es impar, con $x \in (0, \pi)$, $f(x) = \cos(x)$

a) Observe que $(-x) \in (-\pi, 0)$, $f(-x) = -\cos(-x)$, así $f(x) = -f(-x)$. Entonces los coeficientes de la serie son: $A_0 = 0$, $A_n = 0$ y

$$B_n = \frac{2}{\pi}\int_0^{\pi} x\cos(x)sen(nx)dx$$

0,2 pts.

$$\Rightarrow B_n = \frac{-2}{\pi}\left(\frac{\cos((n-1)x)}{2(n-1)} + \frac{\cos((n+1)x)}{2(n+1)}\right)_0^{\pi}$$

$$= \frac{-2}{\pi}\left(\frac{\cos\big((n-1)\pi\big) - 1}{2(n-1)} + \frac{\cos\big((n+1)\pi\big) - 1}{2(n+1)}\right)_0^{\pi} \ ; \ n \neq 1$$

$$= \frac{1}{\pi}(1 - (-1)^{n+1})\left(\frac{2n}{n^2-1}\right) \ ; \ n \neq 1$$

0,2 pts.

Si $n = 1$

$$\Rightarrow B_n == \frac{2}{\pi}\left(\frac{sen(x)^2}{2}\right)_0^{\pi} = 0$$

0,1 pts.

La Serie de Fourier es:

$$\boldsymbol{f(x) = \sum_{n=2}^{\infty} \frac{4}{\pi} \frac{2n}{(4n^2-1)} sen(2nx)}$$

0,2 pts.

b) Para $x_0 = \frac{\pi}{4}$, se tiene convergencia puntual:

$$f\left(\frac{\pi}{4}\right) = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{n}{4n^2-1} sen\left(2n\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{k+1} \frac{8}{\pi} \frac{2k-1}{(4(2k-1)^2-1)} = \frac{\sqrt{2}}{2}$$

$$\sum_{n=1}^{\infty} (-1)^{k+1} \frac{2k-1}{(4(2k-1)^2-1)} = \frac{1}{3} - \frac{3}{35} + \frac{5}{99} \mp \cdots = \frac{\sqrt{2}}{2}\frac{\pi}{8} = \frac{\pi\sqrt{2}}{16}$$

0,6 pts.

c)

$$\frac{2}{\pi}\int_0^{\pi} cos(x)^2 dx = \sum_{n=1}^{\infty} \left(\frac{8}{\pi}\right)^2 \frac{n^2}{(4n^2-1)^2}$$

$$\frac{2}{\pi}\int_0^{\pi} \frac{1+cos(2x)}{2} dx = \sum_{n=1}^{\infty} \frac{64}{\pi^2} \frac{n^2}{(4n^2-1)^2}$$

$$\frac{2}{\pi}\left[\frac{x}{2} + \frac{sen(2x)}{4}\right]_0^{\pi} = \sum_{n=1}^{\infty} \frac{64}{\pi^2} \frac{n^2}{(4n^2-1)^2}$$

$$1 = \sum_{n=1}^{\infty} \frac{64}{\pi^2} \frac{n^2}{(4n^2-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}$$

0,7 pts.

Pregunta 2:

Dada la curva $\vec{\mathbf{r}}(\mathbf{t}) = (\mathbf{t}\cos\mathbf{t}, \mathbf{tsent}, \mathbf{t})$, $\mathbf{t} \geq \mathbf{0}$,

a) determinar el ángulo que forma el vector tangente en cada punto con el eje z.

b) obtener la curvatura y torsión en $\mathbf{P_1} = \left(\mathbf{0}, \frac{\boldsymbol{\pi}}{\mathbf{2}}, \frac{\boldsymbol{\pi}}{\mathbf{2}}\right)$.

c) determinar la ecuación de plano osculador en $\mathbf{P_1} = \left(\mathbf{0}, \frac{\boldsymbol{\pi}}{\mathbf{2}}, \frac{\boldsymbol{\pi}}{\mathbf{2}}\right)$.

Solución:

$$\vec{r}(t) = (t\cos(t), tsen(t), t) \quad \Rightarrow \quad \vec{P}_0 = \vec{r}\left(\frac{\pi}{2}\right) = \left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\vec{r}'(t) = (\cos(t) - tsen(t), sen(t) + tcos(t), 1)$$

$$\vec{r}''(t) = (-sen(t) - sen(t) - \mathrm{t}\cos(\mathrm{t}), \cos(t) + \cos(t) - tsen(t), 0)$$

$$= (-2sen(t) - t\cos(t), 2\cos(t) - tsen(t), 0)$$

Observe que como $\vec{r}'\left(\frac{\pi}{2}\right) = \left(-\frac{\pi}{2}, 1,1\right)$ y $\vec{r}''\left(\frac{\pi}{2}\right) = \left(-2, -\frac{\pi}{2}, 0\right)$

$$\Rightarrow \vec{r}'\left(\frac{\pi}{2}\right) \times \vec{r}''\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}, -2, \frac{\pi^2}{4} + 2\right)$$

$$\vec{r}'''(t) = (-2\cos(t) - \cos(t) + tsen(t), -2sen(t) - sen(t) - tcos(t), 0)$$

$$= (-3\cos(t) + tsen(t), -3sen(t) - tcos(t), 0)$$

$$\Rightarrow \vec{r}'''\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}, -3,0\right)$$

Por otro lado,

$$\left||\vec{r}(t)|\right| = \sqrt{[(\cos(t) - tsen(t))^2 + (sen(t) + tcos(t))^2 + 1]} = \sqrt{2 + \mathrm{t}^2}$$

a) $\vec{r}(t) \cdot \hat{k} = \left||\vec{r}(t)|\right| \left||\hat{k}|\right| \cos(\emptyset) = \sqrt{2 + \mathrm{t}^2}\cos(\Phi) = 1$

$$\Rightarrow \cos(\Phi) = \frac{1}{\sqrt{2 + \mathrm{t}^2}}$$

$$\Rightarrow \boldsymbol{\Phi} = \boldsymbol{arccos}\left(\frac{\mathbf{1}}{\sqrt{\mathbf{2} + \mathbf{t}^2}}\right)$$

b) La curvatura en P_1 es $\hat{k}\left(\frac{\pi}{2}\right) = \frac{\left\|\vec{r}'\left(\frac{\pi}{2}\right) \times \vec{r}''\left(\frac{\pi}{2}\right)\right\|}{\left\|\vec{r}'\left(\frac{\pi}{2}\right)\right\|} = \frac{\sqrt{\frac{\pi^2}{4} + 4 + \left(\frac{\pi^2}{4} + 2\right)^2}}{\left(\frac{\pi^2}{4} + 2\right)^{3/2}} = \frac{\sqrt{\mathbf{8} + \frac{\mathbf{5}\boldsymbol{\pi}^2}{\mathbf{4}} + \frac{\boldsymbol{\pi}^4}{\mathbf{16}}}}{\left(\frac{\boldsymbol{\pi}^2}{\mathbf{4}} + \mathbf{2}\right)^{\mathbf{3/2}}}$

La torsión en P_1 es $\hat{\tau}\left(\frac{\pi}{2}\right) = \frac{\vec{r}'\left(\frac{\pi}{2}\right)\times\vec{r}''\left(\frac{\pi}{2}\right)\cdot\vec{r}'''\left(\frac{\pi}{2}\right)}{\left\|\vec{r}'\left(\frac{\pi}{2}\right)\times\vec{r}''\left(\frac{\pi}{2}\right)\right\|^2} = \frac{\left(\frac{\pi}{2},-2,\frac{\pi^2}{4}+2\right)\cdot\left(\frac{\pi}{2},-3,0\right)}{8+\frac{5\pi^2}{4}+\frac{\pi^4}{16}} = \boldsymbol{\frac{\frac{\pi^2}{4}+6}{8+\frac{5\pi^2}{4}+\frac{\pi^4}{16}}}$

c) La ecuación del plano osculador en P_0 es:

$$\left(\vec{R}-\left(0,\frac{\pi}{2},\frac{\pi}{2}\right)\right)\cdot\vec{r}'\left(\frac{\pi}{2}\right)\times\vec{r}''\left(\frac{\pi}{2}\right)=0$$

$$\Rightarrow \left(\vec{R}-\left(0,\frac{\pi}{2},\frac{\pi}{2}\right)\right)\cdot\left(\frac{\pi}{2},-2,\frac{\pi^2}{4}+2\right)=0$$

$$\Rightarrow \frac{\pi}{2}x-2y+\left(\frac{\pi^2}{4}+2\right)z=-\pi+\frac{\pi^3}{8}+\pi$$

$$\Rightarrow \boldsymbol{\frac{\pi}{2}x-2y+\left(\frac{\pi^2}{4}+2\right)z=\frac{\pi^3}{8}}$$

Pregunta 3:

a) Sea $\mathbf{f(x,y)} = \begin{cases} \dfrac{2xy^2}{x^2+y^4}, & \text{si } (x,y) \neq (0,0) \\ 0, & \text{si } (x,y) = (0,0) \end{cases}$, determinar si **f** es continua en $P_0 = (0, 0)$ y obtener $f_X(0, 0)$ y $f_Y(0, 0)$, si existen.

b) Sea $\mathbf{f(x,y)} = (x^2+y^2)\ln(x^2+y^2)$, **si** $(x,y) \neq (0,0)$. Verificar que f es solución de la ecuación $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 2f(x,y)$.

Solución:

a) Con trayectoria hacia $P_0 = (0,0)$, como:

(i) $y = 0 \Rightarrow f(x,0) = \frac{2x*0}{x^2}$, se aproxima a $L_1 = 0$

(ii) $x = 0 \Rightarrow f(0,y) = \frac{2*0*y}{y^4}$, se aproxima a $L_2 = 0$

0,2 pts.

(iii) $y = x \Rightarrow f(x,x) = \frac{2x^3}{x^2+x^4} = \frac{2x^2x}{x^2(1+x^2)}$, se aproxima a $L_3 = 0$

(También se puede considerar $y = mx$)

(iv) $x = y^2$, (parábola), $f(y^2,y) = \frac{2y^2y^2}{y^4+y^4} = \frac{2y^4}{2y^4} = 1$, se aproxima a $L_4 = 1$

0,2 pts.

∴ Como $L_1 = L_2 = L_3 \neq L_4 \Rightarrow$ No existe $lim \frac{2xy^2}{x^2+y^4}$ cuando $(x,y) \to (0,0)$.

$\Rightarrow$ ***f*** **No es continua en** $\boldsymbol{P_0 = (0,0)}$

0,2 pts.

Las derivadas parciales:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{\frac{2(\Delta x) * 0^2}{\Delta x^2 + 0^2} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{\frac{2 * 0(\Delta y)^2}{0^2 + (\Delta y)^4} - 0}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

0,2 pts.

b)

$$f(x,y) = \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial f}{\partial x} = \frac{y\sqrt{x^2 + y^2} - \frac{(2x)xy}{2\sqrt{x^2 + y^2}}}{\left(\sqrt{x^2 + y^2}\right)^2} = \frac{y^3}{(x^2 + y^2)\sqrt{x^2+y^2}} = y^3\sqrt{x^2+y^2}$$

0,5 pts.

$$\frac{\partial f}{\partial y} = \frac{x\sqrt{x^2 + y^2} - \frac{(2y)xy}{2\sqrt{x^2 + y^2}}}{\left(\sqrt{x^2 + y^2}\right)^2} = \frac{x^3}{(x^2 + y^2)\sqrt{x^2+y^2}} = x^3\sqrt{x^2+y^2}$$

0,5 pts.